Lecture 12: October 7

The goal of today's lecture is to prove Theorem 9.1. Let me first recall the problem. From a polarized variation of Hodge structure (of weight n) on the punctured disk, we had constructed the period mapping $\Phi\colon \tilde{\mathbb{H}} \to D$. We also noted that

$$e^{-zR}\Phi(z) = e^{-zR_S}e^{-zR_N}\Phi(z)$$

is invariant under the substitution $z \mapsto z + 2\pi i$, and therefore descends to a holomorphic mapping $\Psi\colon \Delta^* \to \check{D}$. Now Theorem 9.1 is the statement that Ψ extends holomorphically to the entire disk Δ. What we are actually going to prove is that Ψ extends continuously; this is enough, by Riemann's extension theorem.

More precisely, we are going to prove the following distance estimate:

Proposition 12.1. *There are constants $B, C, \delta, \varepsilon > 0$ such that*

$$d_{\check{D}}\Big(e^{-zR}\Phi(z), e^{-(z+w)R}\Phi(z+w)\Big) \leq C|w|e^{\varepsilon \operatorname{Re} z}$$

holds for every $z \in \tilde{\mathbb{H}}$ with $\operatorname{Re} z < -B$, and every $w \in \mathbb{C}$ with $|w| < \delta$.

You should think of this as saying that the derivative of the mapping $e^{-zR}\Phi(z)$ takes the tangent vector $\frac{\partial}{\partial z}$ to a vector whose length, with respect to the metric $h_{\check{D}}$, is at most $Ce^{\varepsilon \operatorname{Re} z}$; this estimate holds on the halfspace $\operatorname{Re} z < -B$. To keep the notation simple, I have put this derivative bound in terms of distances, but they are clearly equivalent.

Note. One can say more about the dependence of the constants: for period mappings with the property that $\Phi(-1)$ lies in a fixed compact subset of D, the constants in the proposition only depend on the period domain D and the monodromy operator T, but not on the specific period mapping being considered. This is important in the proof of the higher-dimensional version of Schmid's results.

It is straightforward to deduce from Proposition 12.1 that Ψ extends continuously over the origin. Let $t_1, t_2 \in \Delta^*$ be two points with $|t_1| \leq |t_2| < e^{-B}$. Choose preimages $z_1, z_2 \in \tilde{\mathbb{H}}$ such that $t_1 = e^{z_1}$ and $t_2 = e^{z_2}$; these are unique if we specify that $\operatorname{Re} z_1 \leq \operatorname{Re} z_2 < -B$ and $0 \leq \operatorname{Im} z_1, \operatorname{Im} z_2 < 2\pi$. We can estimate the distance

$$d_{\check{D}}\Big(\Psi(t_1), \Psi(t_2)\Big) = d_{\check{D}}\Big(e^{-z_1 R}\Phi(z_1), e^{-z_2 R}\Phi(z_2)\Big)$$

by integrating first along a line segment of length at most 2π (with constant real part $\operatorname{Re} z_2$), and then along the line segment from $\operatorname{Re} z_1$ to $\operatorname{Re} z_2$ (with constant imaginary part $\operatorname{Im} z_2$). Because of the derivative bound in Proposition 12.1, we get

$$\begin{aligned} d_{\check{D}}\Big(\Psi(t_1), \Psi(t_2)\Big) \leq 2\pi \cdot Ce^{\varepsilon \operatorname{Re} z_2} + \int_{\operatorname{Re} z_1}^{\operatorname{Re} z_2} Ce^{\varepsilon x}\, dx &\leq C\left(2\pi + \frac{1}{\varepsilon}\right)e^{\varepsilon \operatorname{Re} z_2} \\ &= C\left(2\pi + \frac{1}{\varepsilon}\right)|t_2|^{\varepsilon}. \end{aligned}$$

This goes to zero independently of t_1, and so Ψ does extend continously over the origin. By construction, we have $\Psi(0) \in \check{D}$.

Outline of the proof. The key ingredient in the proof is the distance-decreasing property of period mappings. Since this only holds for the $G_{\mathbb{R}}$-invariant distance on D, we first need to rephrase the problem in terms of d_D. In

$$d_{\check{D}}\Big(e^{-zR}\Phi(z), e^{-(z+w)R}\Phi(z+w)\Big),$$

we first drop the common factor e^{-zR}; then $e^{-wR}\Phi(z+w)$ is certainly in D as long as $|w|$ is very small, and so it makes sense to consider

$$d_D\Big(\Phi(z), e^{-wR}\Phi(z+w)\Big).$$

Now remember that $D \cong G_{\mathbb{R}}/H$, with the base point $o \in D$ corresponding to the coset H. Since $G_{\mathbb{R}} \to D$ is a fiber bundle, with fiber the compact subgroup H, one can lift the period mapping $\Phi\colon \tilde{\mathbb{H}} \to D$ to a C^∞-mapping $g\colon \tilde{\mathbb{H}} \to G_{\mathbb{R}}$, with the property that $\Phi(z) = g(z)\cdot o$.

Note. Of course, g is only determined up to right multiplication by H. One can show that there is a distinguished lifting g, which is even real-analytic; its properties are studied in depth in Schmid's famous SL_2-orbit theorem.

Anyway, since the distance function d_D is $G_{\mathbb{R}}$-invariant, we have

$$d_D\Big(\Phi(z), e^{-wR}\Phi(z+w)\Big) = d_D\Big(o, g(z)^{-1}e^{-wR}g(z+w)\cdot o\Big).$$

Let me briefly outline how the proof is going to go. We start by investigating for which values of $w \in \mathbb{C}$ the point $g(z)^{-1}e^{-w}g(z+w)\cdot o \in \check{D}$ lies in the period domain D. Initially, it looks like this should only be true when $|w|$ is very small (because it holds at $w = 0$, and D is open in $\check{D}$), but we will use the distance-decreasing property to show that it actually holds on a vertical strip of the form

$$|\mathrm{Re}\, w| < \gamma|\mathrm{Re}\, z|.$$

We will then use the fact that the mapping $e^{-wR}\Phi(z+w)$ is holomorphic in w and invariant under the substitution $w \mapsto w + 2\pi i$ to estimate its derivative at $w = 0$, which gives us a good upper bound for

$$d_{\check{D}}\Big(o, g(z)^{-1}e^{-wR}g(z+w)\cdot o\Big).$$

This is the crucial step; after that, all we need to do is move $g(z)$ over to the other side and put the factor e^{-zR} back. (There are some technical complications at the end, but this is the basic idea.)

Details of the proof. We start by choosing an open neighborhood $o \in U \subseteq D$ isomorphic to a polydisk in $\mathbb{C}^N$. If we make U sufficiently small, we can assume that the distance functions d_D and $d_{\check{D}}$, as well as the Euclidean distance on the polydisk, are mutually bounded up to a constant.

Step 1. The distance-decreasing property of period mappings (Corollary 7.10) gives

$$\begin{aligned} d_D\Big(o, g(z)^{-1}g(z+w)\cdot o\Big) &= d_D\Big(g(z)\cdot o, g(z+w)\cdot o\Big) \\ &= d_D\Big(\Phi(z), \Phi(z+w)\Big) \leq d_{\tilde{\mathbb{H}}}(z, z+w) \leq \frac{C|w|}{|\mathrm{Re}\, z|}, \end{aligned}$$

where the last inequality holds on the vertical strip $|\mathrm{Re}\, w| < \frac{1}{2}|\mathrm{Re}\, z|$, for example. By the triangle inequality,

$$\begin{aligned} &d_D\Big(o, g(z)^{-1}e^{-wR}g(z+w)\cdot o\Big) \\ &\leq d_D\Big(g(z)^{-1}g(z+w), g(z)^{-1}e^{-wR}g(z+w)\cdot o\Big) + d_D\Big(o, g(z)^{-1}g(z+w)\cdot o\Big) \\ &\leq d_D\Big(o, g(z+w)^{-1}e^{-wR}g(z+w)\cdot o\Big) + \frac{C|w|}{|\mathrm{Re}\, z|}, \end{aligned}$$

assuming that all the points in question lie in D, of course. The first term can be estimated using the following lemma.

Lemma 12.2. *There are constants $B, C, r > 0$ such that*

$$d_D\Big(o, g(z+w)^{-1}e^{-wR}g(z+w)\cdot o\Big) \leq \frac{C|w|}{|\operatorname{Re} z|}$$

for every $z \in \tilde{\mathbb{H}}$ with $\operatorname{Re} z < -B$, and every $w \in \mathbb{C}$ with $|w| < r|\operatorname{Re} z|$.

Putting the two things together, we arrive at the inequality

$$d_D\Big(o, g(z)^{-1}e^{-wR}g(z+w)\cdot o\Big) \leq \frac{C|w|}{|\operatorname{Re} z|},$$

which holds for $\operatorname{Re} z < -B$ and $|w| < r|\operatorname{Re} z|$. Shrinking r, if necessary, we can therefore arrange that

$$g(z)^{-1}e^{-wR}g(z+w)\cdot o \in U \subseteq D$$

as long as $\operatorname{Re} z < -B$ and $|w| < r|\operatorname{Re} z|$. After further increasing the value of B, we can arrange moreover that the set

$$\big\{\, w \in \mathbb{C} \;\big|\; |w| < r|\operatorname{Re} z| \,\big\}$$

contains the rectangular box

$$\big\{\, w \in \mathbb{C} \;\big|\; |\operatorname{Re} w| < \gamma|\operatorname{Re} z| \text{ and } 0 \leq \operatorname{Im} w \leq 2\pi \,\big\},$$

where $\gamma = \frac{1}{2}r$, say. Now remember that

$$g(z)^{-1}e^{-wR}g(z+w)\cdot o = g(z)^{-1}e^{-wR}\Phi(z+w)$$

is invariant under $w \mapsto w + 2\pi i$. This means that if $g(z)^{-1}e^{-wR}g(z+w) \in U$ for every w in a box of height 2π, then the same thing is true on the whole vertical strip $|\operatorname{Re} w| < \gamma|\operatorname{Re} z|$. We can summarize the result of the first step as follows: there are constants $B, \gamma > 0$ such that

$$g(z)^{-1}e^{-wR}g(z+w)\cdot o \in U \tag{12.3}$$

for every $z \in \tilde{\mathbb{H}}$ with $\operatorname{Re} z < -B$, and every $w \in \mathbb{C}$ with $|\operatorname{Re} w| < \gamma|\operatorname{Re} z|$.

Step 2. Recall that U is ismomorphic to a polydisk in $\mathbb{C}^N$. Each of the N coordinate functions, applied to the point

$$g(z)^{-1}e^{-wR}g(z+w)\cdot o = g(z)^{-1}e^{-wR}\Phi(z+w),$$

is therefore a holomorphic function of w that is bounded, defined on the vertical strip $|\operatorname{Re} w| < \gamma|\operatorname{Re} z|$, and periodic of period $2\pi i$. The following cute lemma, due to Schmid and Deligne, provides an upper bound on the derivative of such a function. (This is an instance of the general principle that, in order for a holomorphic function to be defined on a big neighborhood of a given point, its Taylor coefficients at that point must be small.)

Lemma 12.4. *Let f be a holomorphic function that is bounded, defined on a vertical strip of the form $|\operatorname{Re} w| < \gamma x$, and periodic of period $2\pi i$. Then*

$$|f'(0)| \leq 4\pi \cdot \frac{e^{\gamma x}}{(e^{\gamma x}-1)^2} \cdot \sup\big\{\, |f(w)| \;\big|\; |\operatorname{Re} w| < \gamma x \,\big\}.$$

Proof. The fact that f is periodic implies that $f(w) = g(e^w)$, where

$$g\colon \big\{\, t \in \mathbb{C} \;\big|\; e^{-\gamma x} < t < e^{\gamma x} \,\big\} \to \mathbb{C}$$

is a bounded holomorphic function defined on an annulus. Since $f'(0) = g'(1)$, it suffices to estimate the derivative $g'(1)$; this can be done using the residue theorem. For $\varepsilon > 0$ sufficiently small, the residue theorem gives

$$g'(1) = \int_{|t|=e^{\gamma x-\varepsilon}} \frac{g(t)\,dt}{(t-1)^2} - \int_{|t|=e^{-(\gamma x-\varepsilon)}} \frac{g(t)\,dt}{(t-1)^2},$$

and after using the triangle inequality and doing some easy integrals, we arrive at

$$|g'(1)| \leq 4\pi \cdot \frac{e^{\gamma x-\varepsilon}}{(e^{\gamma x-\varepsilon}-1)^2} \cdot \sup\left\{ |g(t)| \;\middle|\; e^{-(\gamma x-\varepsilon)} < |t| < e^{\gamma x-\varepsilon} \right\}.$$

Now let $\varepsilon \to 0$ to get the desired inequality for $|f'(0)| = |g'(1)|$. □

As long as x is sufficiently large, we have

$$\frac{e^{\gamma x}}{(e^{\gamma x}-1)^2} \leq 2e^{-\gamma x},$$

which is the sort of upper bound we are looking for. Back to our problem. Lemma 12.4, applied to the coordinates (with respect to the polydisk) of the point

$$g(z)^{-1}e^{-wR}g(z+w)\cdot o = g(z)^{-1}e^{-wR}\Phi(z+w),$$

gives us an upper bound on the derivative at $w = 0$. If we phrase this is terms of distances, it says that there are constants $B, C, \gamma, \delta > 0$, such that

$$d_{\check{D}}\Big(o, g(z)^{-1}e^{-wR}g(z+w)\cdot o\Big) < C|w| \cdot e^{\gamma \operatorname{Re} z} \tag{12.5}$$

for every $z \in \tilde{\mathbb{H}}$ with $\operatorname{Re} z < -B$, and every $w \in \mathbb{C}$ with $|w| < \delta$. (Here we are using the fact that the distance function $d_{\check{D}}$ on U, and the Euclidean distance on the polydisk, are mutually bounded up to a constant.)

Step 3. It remains to put everything back into the right place. The following lemma allows us to more $g(z)$ back to the first argument.

Lemma 12.6. *There is are constants $B, C > 0$ and an integer $\ell \in \mathbb{N}$ such that*

$$\|\operatorname{Ad} g(z)\| \leq C|\operatorname{Re} z|^{\ell}$$

for every $z \in \tilde{\mathbb{H}}$ with $\operatorname{Re} z < -B$.

Combining this lemma with Lemma 11.5, we deduce from (12.5) that

$$d_{\check{D}}\Big(\Phi(z), e^{-wR}\Phi(z+w)\Big) = d_{\check{D}}\Big(g(z)\cdot o, e^{-wR}g(z+w)\cdot o\Big) < C|w| \cdot |\operatorname{Re} z|^{\ell} e^{\gamma \operatorname{Re} z}.$$

After increasing B and slightly shrinking γ, we can put this back into the form

$$d_{\check{D}}\Big(\Phi(z), e^{-wR}\Phi(z+w)\Big) < C|w| \cdot e^{\gamma \operatorname{Re} z}, \tag{12.7}$$

again valid for every $z \in \tilde{\mathbb{H}}$ with $\operatorname{Re} z < -B$, and every $w \in \mathbb{C}$ with $|w| < \delta$.

Step 4. The last thing is to put back the factor e^{-zR}. Since $e^{-zR}\Phi(z)$ is invariant under $z \mapsto z + 2\pi i$, we can restrict to points $z \in \tilde{\mathbb{H}}$ with $0 \leq \operatorname{Im} z \leq 2\pi$. Now

$$e^{-zR} = e^{-\operatorname{Re} z R_S} e^{-\operatorname{Re} z R_N} e^{-i \operatorname{Im} z R_S} e^{-i \operatorname{Im} z R_N},$$

and the third and fourth factor are obviously bounded as long as $0 \leq \operatorname{Im} z \leq 2\pi$. Furthermore, R_N is nilpotent, and so

$$\|\operatorname{Ad} e^{-\operatorname{Re} z R_N}\| \leq C|\operatorname{Re} z|^{\ell}$$

for a suitable constant $C > 0$ and integer $\ell \in \mathbb{N}$. This is neglible compared to the exponential in our estimate, and so the factor $e^{-\operatorname{Re} z R_N}$ is harmless. What about the remaining factor $e^{-\operatorname{Re} z R_S}$? Recall from Lemma 11.6 that

$$\|\operatorname{Ad} e^{-\operatorname{Re} z R_S}\| \leq C e^{(\alpha_{max}-\alpha_{min}) \operatorname{Re} z},$$

where α_{max} and α_{min} are the largest and smallest eigenvalues of R_S. Set $\rho = \alpha_{max} - \alpha_{min}$; this is a real number in the interval $[0, 1)$. Putting everything together, and adjusting B and γ as before, we find that

$$d_{\check{D}}\Big(e^{-zR}\Phi(z), e^{-(z+w)R}\Phi(z+w)\Big) < C|w| \cdot e^{\gamma \operatorname{Re} z} \cdot e^{\rho|\operatorname{Re} z|}, \tag{12.8}$$

valid for every $z \in \tilde{\mathbb{H}}$ with $\operatorname{Re} z < -B$, and every $w \in \mathbb{C}$ with $|z| < \delta$. Here we run into a serious problem: the difference $\rho = \alpha_{max} - \alpha_{min}$ may well be bigger than the small number $\gamma > 0$, and so putting back the factor $e^{-\operatorname{Re} z R_S}$ has ruined our estimate. Since there is no way to increase the value of γ, it looks at first glance as if we are doomed.

Step 5. Fortunately, there is a way around this nasty problem. Namely, as I already suggested at the end of Lecture 11, we can use cyclic coverings to squeeze the eigenvalues of R_S closer together. In order not to make the notation confusing, we are going to work entirely on the halfspace $\tilde{\mathbb{H}}$ though – the cyclic coverings will only happen implicitly.

Recall that $T = e^{2\pi i R_N} e^{2\pi i R_S}$, where R_S is semisimple with eigenvalues in a fixed interval I. For any $m \geq 1$, we can pick a semisimple operator $S_m \in \operatorname{End}(V)$, with eigenvalues in the interval $[-\frac{1}{2m}, \frac{1}{2m})$, such that

$$e^{2\pi i m R_S} = e^{2\pi i m S_m}.$$

With this choice, mS_m has eigenvalues in the fixed interval $[-\frac{1}{2}, \frac{1}{2})$. Note that S_m and R_S have the same eigenspaces (but with different eigenvalues); in particular, each S_m commutes with R_N. Now consider the expression

$$g(z)^{-1} e^{-w(R_N + S_m)} \Phi(z + w) \in \check{D}.$$

It is still holomorphic, but only invariant under the substitution $w \mapsto w + 2\pi i m$. By applying our previous analysis to this function, we get

$$d_{\check{D}}\Big(e^{-z(R_N+S_m)}\Phi(z), e^{-(z+w)(R_N+S_m)}\Phi(z+w)\Big) < C|w| \cdot e^{\frac{\gamma}{m}\operatorname{Re} z} \cdot e^{\rho_m |\operatorname{Re} z|}, \tag{12.9}$$

where ρ_m is the difference between the largest and smallest eigenvalues of S_m. The additional $\frac{1}{m}$ in the exponent comes from adapting Lemma 12.4 to holomorphic functions that are periodic of period $2\pi i m$.

Step 6. This still doesn't look good: we can move the eigenvalues of S_m closer together by increasing m, but only at the cost of replacing γ by the much smaller number $\frac{\gamma}{m}$. Fortunately, this problem can be solved with the help of results in *Diophantine approximation*. Here is why. Suppose that α is one of the eigenvalues of R_S. It is easy to find the corresponding eigenvalue of S_m: this is

$$\frac{m\alpha - k}{m} = \alpha - \frac{k}{m},$$

where k is the integer closest to $m\alpha$. We are trying to get ρ_m, the difference between the largest and smallest eigenvalue of S_m, to be less than $\frac{2\gamma}{3m}$, say, and so we need an inequality of the form

$$\left|\alpha - \frac{k}{m}\right| \leq \frac{\gamma}{3m}.$$

This is clearly a problem in Diophantine approximation, which is solved by the following basic result due to Peter Gustav Lejeune Dirichlet, called the *Dirichlet approximation theorem.*

Theorem 12.10. *For any real numbers $\alpha_1, \dots, \alpha_d \in \mathbb{R}$, and for every $n \geq 1$, there exists an integer q with $1 \leq q \leq n^d$, and integers $p_1, \dots, p_d \in \mathbb{Z}$, such that*

$$\left|\alpha_i - \frac{p_i}{q}\right| \leq \frac{1}{qn}$$

for every $i = 1, \dots, d$.

Proof. The proof is a nice exercise in the use of the pigeonhole principle (which Dirichlet invented for this purpose, originally calling it the "box principle"). For any real number $\alpha \in \mathbb{R}$, denote by $\{\alpha\} \in [0,1)$ the fractional part. Divide the d-dimensional box $[0,1]^d$ into n^d smaller boxes of side length $\frac{1}{n}$, in the obvious way. For $k = 0, 1, \dots, n^d$, consider the vector

$$\Big(\{k\alpha_1\}, \dots, \{k\alpha_d\}\Big) \in [0,1]^d.$$

Since there are $n^d + 1$ vectors, but only n^d boxes, two vectors have to land in the same box. This gives us two integers k and $k+q$, with $1 \leq q \leq n^d$, such that

$$\Big|\{(k+q)\alpha_i\} - \{k\alpha_i\}\Big| \leq \frac{1}{n}$$

for every $i = 1, \dots, n$. This says that there are integers $p_1, \dots, p_d \in \mathbb{Z}$ such that

$$|q\alpha_i - p_i| \leq \frac{1}{n},$$

which is equivalent to the desired inequality. □

In order to apply this to our setting, let $d = \dim V$. If we take $n \geq \frac{\gamma}{3}$, then Dirichlet's approximation theorem guarantees the existence of an integer m with $1 \leq m \leq n^d$, such that all eigenvalues of S_m have absolute value at most $\frac{\gamma}{3m}$, and therefore $\rho_m \leq \frac{2\gamma}{3m}$. We then get

$$d_{\check{D}}\Big(e^{-z(R_N+S_m)}\Phi(z), e^{-(z+w)(R_N+S_m)}\Phi(z+w)\Big) < C|w| \cdot e^{\varepsilon \operatorname{Re} z}, \tag{12.11}$$

where $\varepsilon = \frac{\gamma}{3m}$ is now unbelievably tiny, but still positive. By the Riemann extension theorem, this is still enough to ensure that the holomorphic mapping

$$\Psi_m \colon \Delta^* \to \check{D},$$

defined by the condition that

$$\Psi_m(e^{\frac{z}{m}}) = e^{-z(R_N+S_m)}\Phi(z),$$

extends holomorphically over the origin. According to Lemma 11.7, this suffices to conclude that our original mapping Ψ also extends holomorphically over the origin. This proves Theorem 9.1.